Quiz 4

Based on iterative compression.

Duration: 60 minutes | Total Marks: 20

Problem 1. Total: 10

In d-**Hitting Set**, we are given a universe S, a family C of subsets of size at most d, and a budget k. The task is to decide whether there is a hitting set $H \subseteq S$ with $|H| \leq k$. These questions explore iterative compression assuming a solver for $(d-1)$-Hitting Set.

(1.a) (2 points) Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be a set of elements, and let C be the collection of subsets:

$$C = \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{1, 4, 5\}, \{2, 6, 7\}, \{1, 7, 8\}, \{3, 8, 9\}, \{2, 5, 9\}, \{1, 6, 9\}, \{3, 6, 8\}, \{2, 4, 8\}, \{1, 5, 7\}\}$$

Now, suppose we are given $X = \{1, 2, 3, 7\}$ as a Hitting Set of size 4 and we want to know if there is a hitting set Y of size 3. Let us assume that $Y \cap X = \{1, 2\}$. Then we can throw away the sets that contain 1 or 2 from C. What is the collection that remains?

A. $\{\{3, 4, 5\}, \{2, 6, 7\}, \{3, 4, 8\}, \{3, 5, 7\}\}$

B. $\{\{3, 4, 5\}, \{3, 6, 7\}, \{3, 7, 8\}\}$

C. $\{\{3, 4, 5\}, \{3, 8, 9\}, \{3, 6, 8\}\}$

D. $\{\{3, 4, 5\}, \{3, 8, 9\}\}$

(1.b) (2 points) Continuing from the example in the previous question, recall that we have $Y \cap X = \{1, 2\}$. Now, we want to know if there is a hitting set Y of size 3 overall. This will be the case if and only if:

A. The family that remains after throwing away sets that contain either 1 or 2 must have a hitting set of size 1.

B. The family that remains after throwing away sets that contain either 1 or 2 must have a hitting set of size 1 that does not contain either 3 or 7.

C. The family that remains after throwing away sets that contain either 1 or 2 must have a hitting set of size 1 that contains at most one of 3 or 7.

(1.c) (2 points) Based on your answer to the previous question, simplify the leftover family from the first question to observe that solving the remaining problem is now equivalent to:

A. Vertex Cover B. d-Hitting Set on a smaller family C. $(d-1)$-Hitting Set on a smaller family D. $(d+1)$-Hitting Set on a smaller family

(1.d) (2 points) For the example in the first question, we can extend the solution $\{1, 2\}$ to a complete solution of size at most three if and only if:

A. The sets $\{4, 5\}$, $\{8, 9\}$ and $\{6, 8\}$ have a single common element.

B. The sets $\{4, 5\}$, $\{8, 9\}$ and $\{6, 8\}$ have a hitting set of size at most two

(1.e) (2 points) If we know how to solve $(d-1)$-Hitting Set in time $O(c^k \cdot \text{poly}(n))$, then we can solve d-Hitting Set in time:

A. $O(c^k \cdot \text{poly}(n))$ B. $O((c-1)^k \cdot \text{poly}(n))$ C. $O((c+1)^k \cdot \text{poly}(n))$

Problem 2. Total: 10

In **Odd Cycle Transversal (OCT)**, we are given an undirected graph G and a positive integer k. The goal is to find a set $X \subseteq V(G)$ with $|X| \leq k$ such that $G - X$ is bipartite, or to conclude that no such set exists. These questions walk through the key ideas behind solving OCT using iterative compression.

Suppose we apply iterative compression and arrive at the following subroutine, called **Disjoint Odd Cycle Transversal**: we are given G, an integer k, and a set W of $k + 1$ vertices such that $G - W$ is bipartite. The objective is to find a set $X \subseteq V(G) \setminus W$ of at most k vertices such that $G - X$ is bipartite, or to conclude that no such set exists.

(2.a) (1 point) Since $X \subseteq V(G) \setminus W$, every vertex of W remains in $G - X$. In particular, $G[W]$ must also be bipartite. Suppose $G[W]$ is indeed bipartite and has c connected components. How many proper 2-colorings $f_W : W \to \{1, 2\}$ does $G[W]$ admit?

A. c B. $k + 1$ C. 2^c D. 2^{k+1}

(2.b) (1 point) For a fixed proper 2-coloring f_W of $G[W]$, let $B_1^W = f_W^{-1}(1)$ and $B_2^W = f_W^{-1}(2)$. Consider a vertex $v \in V(G) \setminus W$ that is adjacent to at least one vertex in B_1^W and at least one vertex in B_2^W. What can we conclude about such a vertex v? It is important to note that this question is in the context of a fixed choice of f_W, which is our guess for how the vertices of W get colored in $G - X$.

A. v must be colored 1 in $G - X$

B. v must be colored 2 in $G - X$

C. v can be colored either 1 or 2

D. v must be included in X (i.e., deleted)

(2.c) (2 points) Assume there are no vertices adjacent to both color classes of W. Then the remaining vertices of $V(G) \setminus W$ fall into three categories: those with neighbors only in B_1^W (call this set B_2, since they must receive color 2 or be deleted), those with neighbors only in B_2^W (call this set B_1, since they must receive color 1 or be deleted), and those with no neighbors in W at all.

Independently, recall that $G - W$ is bipartite and already has some proper 2-coloring. Let us denote this 2-coloring by $f^* : V(G - W) \to \{1, 2\}$.

Such an instance — a bipartite graph with a known 2-coloring f^* and prescribed color classes B_1, B_2 — the problem is called **Annotated Bipartite Coloring**. Consider the vertices of $C := (B_1 \cap (f^*)^{-1}(2)) \cup (B_2 \cap (f^*)^{-1}(1))$, i.e., those that must *change* their color relative to f^*, and the vertices of $R := (B_1 \cap (f^*)^{-1}(1)) \cup (B_2 \cap (f^*)^{-1}(2))$, i.e., those that must *retain* their color.

We don't know X yet, but for analysis, we observe that any connected component of $G \setminus X$, either all vertices keep their f^*-color or all vertices flip (why?). Thus, which of the following must hold?

A. Every vertex of C must be included in X

B. Every vertex of R must be included in X

C. No connected component of $G \setminus X$ contains both a vertex of C and a vertex of R

D. $|C| \leq |R|$

(2.d) (2 points) Based on your answer to the previous question, how will you find the set X?

A. A greedy algorithm that iteratively removes high-degree vertices

B. A maximum flow / minimum cut computation

C. Dynamic programming on a tree decomposition

D. A reduction to 2-SAT

(2.e) (2 points) Based on your choices above, what is your running time for the **Disjoint** Odd Cycle Transversal problem?

A. $\mathcal{O}(k(n+m))$ B. $\mathcal{O}(2^k \cdot k(n+m))$ C. $\mathcal{O}(k^2(n+m))$ D. $\mathcal{O}(2^{k^2} \cdot (n+m))$

(2.f) (1 point) So far we have solved the **disjoint** variant, where $X \cap W = \emptyset$. For the full compression step, we do not require X to be disjoint from W. Instead, for each vertex $w \in W$, there are three possibilities: (i) $w \in X$ (deleted), (ii) $w \notin X$ and is colored 1, or (iii) $w \notin X$ and is colored 2. For a fixed choice of which vertices of W are deleted, the remainder is a Disjoint OCT instance (with a smaller W and adjusted budget). The total number of such choices across all vertices of W is:

A. 2^{k+1} B. 3^{k+1} C. $(k+1)!$ D. $\binom{2(k+1)}{k+1}$

(2.g) (1 point) Combining everything: each of the 3^{k+1} scenarios leads to an Annotated Bipartite Coloring instance solvable in $\mathcal{O}(k(n+m))$ time, giving $\mathcal{O}(3^k \cdot k(n+m))$ per compression step. The iterative compression framework processes vertices one at a time (n iterations total). What is the overall running time for Odd Cycle Transversal?

A. $\mathcal{O}(2^k \cdot kn(n+m))$ B. $\mathcal{O}(3^k \cdot kn(n+m))$ C. $\mathcal{O}(4^k \cdot n(n+m))$ D. $\mathcal{O}(2^{k^2} \cdot n^2)$